

THE HOROFUNCTION BOUNDARY OF THE LAMPLIGHTER GROUP L_2 WITH THE DIESTEL-LEADER METRIC

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ABSTRACT. We fully describe the horofunction boundary $\partial_h L_2$ with the word metric associated with the generating set $\{t, at\}$ (i.e the metric arising in the Diestel-Leader graph $DL(2, 2)$). The visual boundary $\partial_\infty L_2$ with this metric is a subset of $\partial_h L_2$. Although $\partial_\infty L_2$ does not embed continuously in $\partial_h L_2$, it naturally splits into two subspaces, each of which is a punctured Cantor set and does embed continuously. The height function on $DL(2, 2)$ provides a natural stratification of $\partial_h L_2$, in which countably-many non-Busemann points interpolate between the two halves of $\partial_\infty L_2$. Furthermore, the height function and its negation are themselves non-Busemann horofunctions in $\partial_h L_2$ and are global fixed points of the action of L_2 .

1. INTRODUCTION

The horofunction boundary $\partial_h X$ of a proper complete metric space (X, d) is in general defined as a subspace of the quotient of $C(X)$, the space of continuous \mathbb{R} -valued functions on X , by constant functions [1, Definition II.8.12]. It suffices to choose a base point b in X and use the embedding $i : X \hookrightarrow C(X)$ sending $z \in X \mapsto d(z, x) - d(z, b)$. Since X is proper, the closure \overline{X} of $i(X)$ in $C(X)$ provides a compactification of X . We define $\partial_h X$ to be $\overline{X} \setminus i(X)$. We call a point in \overline{X} a *horofunction*, and given a sequence (y_n) of points in X , one can define a horofunction associated to (y_n) by

$$(1) \quad h_{y_n}(x) = \lim_{n \rightarrow \infty} d(y_n, x) - d(y_n, b)$$

provided this limit exists.

Gromov defines the horofunction boundary, which he calls the *ideal boundary*, in the context of hyperbolic manifolds [5], but the definition applies to any complete metric space. In [1] Bridson and Haefliger use this construction in the context of $CAT(0)$ spaces as a functorial construction of the visual boundary. The horofunction boundary also naturally arises in the study of group C^* -algebras, where Rieffel, referring to it as the *metric boundary*, demonstrates its usefulness particularly in determining the C^* -algebra he calls the *cosphere algebra* [10, §3].

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In this paper, X is a group with a word metric, which is \mathbb{N} -valued.¹ In this setting, we define a geodesic ray to be an isometric embedding $\mathbb{N} \rightarrow X$. We refer to point of $\partial_h X$ as a *Busemann point* if it corresponds to a sequence of points lying along a geodesic ray. We will refer to the space of asymptotic classes of geodesic rays in (X, d) as the *visual boundary* $\partial_\infty X$. In CAT(0) spaces, all horofunctions correspond to Busemann points; in fact, we can extend i to $\bar{i} : X \sqcup \partial_\infty X \rightarrow \bar{X}$, and this is a homeomorphism [1, §II.8.13]. In general one cannot expect an injective, surjective, or even continuous map from $\partial_\infty X$ to $\partial_h X$. Rieffel brings up the question of determining for a given space (X, d) which points of $\partial_h X$ are Busemann points [10, after Definition 4.8]. As an interesting example of non-injectivity, Reiffel demonstrates that there are no non-Busemann points in $\partial_h \mathbb{Z}^n$ with the ℓ_1 norm, and there are countably many Busemann points [10]. However, Kitzmiller and Rathbun demonstrate that $\partial_\infty \mathbb{Z}^n$ is uncountable [7].

Others have studied the horofunction boundary of Cayley graphs of non-CAT(0) groups, often with variation in their terminology, though examples are still sparse.² Develin extended Rieffel's work to abelian groups (he refers to the horofunction boundary as a *Cayley compactification* of the group) [2]. Friedland and Freitas found explicit formulas for horofunctions for $GL(n, \mathbb{C})/U_n$ with Finsler p -metrics (they use the term *Busemann compactification*) [3]. Webster and Winchester (using the term *metric boundary* as Rieffel) studied the action of a word hyperbolic group on its horofunction boundary and found it is amenable [14]. They also established necessary and sufficient conditions for an infinite graph to have non-Busemann points in its horofunction boundary [15]. Walsh has considered the horofunction boundaries of Artin groups of dihedral type [12] and the action of a nilpotent group on its horofunction boundary [13]. Klein and Nikas have studied the horofunction boundary of the Heisenberg group equipped with different metrics [8], [9]. They determine the isometry group of the Heisenberg group with the Carnot-Carathéodory metric.

The *lamplighter group*, discussed more fully at the start of §2, is given by the presentation:

$$L_2 = \langle a, t \mid a^2, [a^{t^i}, a^{t^j}] \forall i, j, \in \mathbb{Z} \rangle$$

Let $S = \{t, at\}$. The generating set S naturally arises when viewing the lamplighter group as a group generated by a finite state automaton (FSA) [4]. This is a rare case where we are able to understand the Cayley graph of such a group with its FSA generating set. In this case, the Cayley graph is the Diestel-Leader graph $DL(2, 2)$ [16]. In [6], the authors describe the visual boundary for Diestel-Leader graphs, which are certain graphs arising from products of regular trees. When there are more than two trees, the topology is indiscrete, but for two trees, the graph inherits enough structure from its component trees that its visual boundary is an interesting non-Hausdorff space. Since $DL(2, 2)$ (the product of two trees with valence 3) is a Cayley graph for L_2 , this provides a boundary for L_2 which is dependent on the generating set. This boundary has a natural partition into two uncountable subsets, which we refer to as the upper and lower visual boundaries and denote by $\partial_\infty L_2^+$ and $\partial_\infty L_2^-$. When equipped with the subspace topology, these subsets are Hausdorff.

¹For us, \mathbb{N} contains 0.

²Note, for non-CAT(0) groups, ∂_h depends on the generating set, as is demonstrated in [10, Example 5.2].

The goal of this paper is to fully describe $\partial_h L_2$ where the metric on L_2 is the word metric from S . In Section 2 we provide some background on this metric, and in Section 3 we discuss the relationship between $\partial_\infty L_2$ and $\partial_h L_2$, proving

Theorem (A - Corollary 3.3 and Observations 3.4 and 6.2). *There is a natural map $\partial_\infty L_2 \rightarrow \partial_h L_2$, which is injective but not continuous. When restricted to either $\partial_\infty L_2^+$ or $\partial_\infty L_2^-$, however, this injection is continuous.*

In Section 4, we explicitly compute formulas for families of horofunctions, including Busemann functions. It turns out the natural height map $H : L_2 \rightarrow \mathbb{Z}$ (see Def. 2.1) is a non-Busemann horofunction. Section 5 provides a proof that all of the points in $\partial_h L_2$ are members of the families described in Section 4, which is our main result.

Theorem (B - Corollary 5.15). *Every point in $\partial_h L_2$ belongs to one of the following families of horofunctions, all of whose formulas we explicitly calculate in Section 4.*

- *Busemann:* These horofunctions arise from certain sequences of lamp stands where the union of positions of lit lamps is bounded below or above and the position of the lamplighter limits to positive or negative infinity.
- *spine:* These horofunctions arise from certain sequences of lamp stands where the union of lit lamps is neither bounded below nor above and the position of the lamplighter limits to a finite value.
- *ribs:* These horofunctions arise from certain sequences of lamp stands where the union of positions of lit lamps is bounded below or above but not both and the position of the lamplighter limits to a finite value.
- *height:* The natural height function and its negation arise as horofunctions from certain sequences of lamp stands where the union of lit lamps is neither bounded below nor above and the position of the lamplighter limits to positive or negative infinity.

The spine is parametrized by \mathbb{Z} and the ribs by a subset of L_2 , and so the set of non-Busemann horofunctions is countable.

We describe the topology of $\partial_h L_2$ in Section 6 by determining the accumulation points, leading to the visualization in Figure 1.

The names of the spine and ribs families come from the topology. The spine family is parametrized by the limiting position of the lamplighter and appears in Figure 1 as the central column of points. For each spine function, there exist two subfamilies of ribs— a “positive rib” and a “negative rib”— each a countable discrete subspace with the spine function as its only accumulation point. See the discussion in Section 4.2 for a thorough description of these subfamilies.

Finally, Section 7 deals with some properties of the natural action of L_2 on $\partial_h L_2$, in particular noting that $\pm H$ are global fixed points.

2. THE DIESTEL-LEADER METRIC ON L_2

Let d denote the word metric on L_2 with generating set $S = \{t, at\}$. Since this is the metric on L_2 induced by the Cayley graph $\text{DL}(2, 2)$, we refer to it as the *Diestel-Leader metric* on L_2 . Whenever we refer to $\partial_\infty L_2$ or $\partial_h L_2$, we always mean with d . Stein and Taback have calculated the metric for general Diestel-Leader graphs [11], but in our case it is simple enough to review and provide a proof.

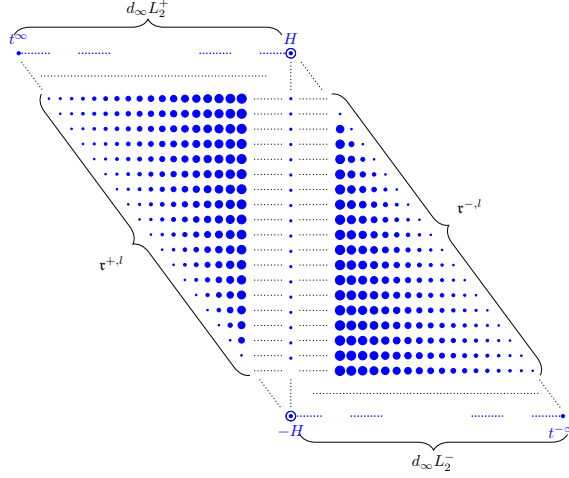


FIGURE 1. Visualization of the horoboundary, including the spine (central column), ribs (discrete point sets limiting to the corresponding spine point), and upper and lower visual boundaries. For each rib, the dots of increasing size represent finite discrete sets whose cardinalities double as we approach the spine.

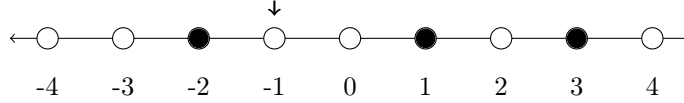


FIGURE 2. A typical element of L_2 .

Each element of L_2 is associated with a “lamp stand”, which consists of an infinite row of lamps in bijective correspondence with \mathbb{Z} , finitely many of which are lit, and a marked lamp indicating the position of the lamplighter. Figure 2 illustrates a typical example. The lamps are binary: either on or off. Right multiplying by a toggles the lamp at the lamplighter’s position, while right multiplying by t increments the position of the lamplighter. We think of this increment as a “step right” as in the figure. Using S , the actions are either “step (right or left)” for $t^{\pm 1}$, “toggle then step right” for at , or “step left then toggle” for $(at)^{-1} = t^{-1}a$.

Definition 2.1. For $g \in L_2$, we define $H(g)$ to be the position of the lamplighter in the lamp stand representing g , or equivalently the exponent sum of t in a word representing g , or the height of g in $DL(2, 2)$.

We define $m(g)$ to be equal to the minimum position of a lit lamp in the lamp stand representation of g if the set of lit lamps is non-empty, and equal to $+\infty$ otherwise. Similarly, we define $M(g)$ to be equal to the maximum position of a lit lamp in the lamp stand representation of g if the set of lit lamps is non-empty, and equal to $-\infty$ otherwise.

For $g_1, g_2 \in L_2$, we define $m(g_1, g_2)$ to be the minimum position of a lamp whose status differs in the lamp stands of g_1 and g_2 if such a position exists, and equal to $+\infty$ otherwise. Similarly, $M(g_1, g_2)$ is the maximum position of a lamp whose

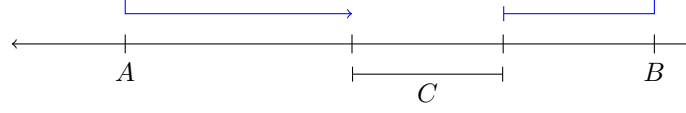


FIGURE 3. Distance between two elements of L_2 with Diestel-Leader metric.

status differs in the lamp stands of g_1 and g_2 if such a position exists, and is $-\infty$ otherwise.

We will define “infinite lamp stands” to represent boundary elements. For these lamp stands, we define the H , m , and M notation analogously.

Lemma 2.2. *If $g_1, g_2 \in L_2$, then $d(g_1, g_2) = 2(B - A) - C$ where*

- $A = \min\{m(g_1, g_2), H(g_1), H(g_2)\}$ *is the left-most position the lamplighter must visit to change between g_1 and g_2 ,*
- $B = \max\{M(g_1, g_2) + 1, H(g_1), H(g_2)\}$ *is the right-most such position, and*
- $C = |H(g_2) - H(g_1)|$ *is the distance between the lamplighter's positions in g_1 and g_2*

See Figure 3 for an illustration of a typical path.

Proof. Since the Cayley graph is vertex transitive, without loss of generality we may assume that $g_1 = id$ and we denote g_2 simply by g . We will consider a geodesic from id to g on the lamp stand representations of the elements of L_2 .

A geodesic will start at id with no lit lamps and the lamplighter at position $H(id) = 0$. The lamplighter must move in one direction (either left or right) until it has gone as far as it needs to, it then travels to the other extremal position, and then finishes by moving to $H(g)$. The initial direction will be *away* from $H(g)$ in order to minimize the total distance. Notice that the minimum extremal position is given by A , which in this case is $A = \min\{m(g), H(g), 0\}$, and the maximal extremal position is given by B , which in this case is $B = \max\{M(g) + 1, H(g), 0\}$. Notice that we use $M(g) + 1$ and not $M(g)$ since to turn on the lamp at position k , the lamplighter must be at position $k + 1$ either immediately before turning on lamp k (if using generator $(at)^{-1}$) or immediately after (if using generator at).

Thus, the second of the three segments of the geodesic will have length $B - A$. The lengths of the first and third segments will sum to less than $B - A$, and the amount less will be exactly equal to the distance between the starting and ending position, which in our case is $|H(g)|$. \square

3. BUSEMANN POINTS

3.1. The visual boundary. As in [6, Section 3.3], we can interpret elements of the visual boundary in terms of the lamp stand model. Such an element can be represented by a geodesic ray emanating from the identity which follows a sequence of steps wherein the lamplighter first moves one direction until reaching the extremal lit lamp in that direction then “turns around” and marches off towards $\pm\infty$ toggling lamps as necessary. Thus, in the limit there is either a minimal lit lamp (if any are lit at all), and the lamplighter is at $+\infty$; or there is a maximal lit lamp (if any are lit at all), and the lamplighter is at $-\infty$. A “turning around” only occurs if the

minimal lit lamp has negative index in the former case, or the maximal lit lamp has positive index in the latter. This final configuration of lit lamps gives an “infinite lamp stand” for the geodesic ray.

In [6, Observations 4.10 and 4.11] the authors investigate the visual boundary of $DL(2, 2)$ and find that as a set, it is a disjoint union of the sets $\partial_\infty L_2^+$ and $\partial_\infty L_2^-$, where $\partial_\infty L_2^\pm$ is the set of those asymptotic classes with lamplighter at $\pm\infty$. These two sets both have the subset topology of punctured Cantor sets, but the full visual boundary is not Hausdorff. We provide the intuition here.

By [6, Lemma 3.5] in L_2 geodesic rays that are asymptotic eventually merge. For example, if a ray has the lamplighter go from 0 to $-n$ and then in the positive direction forever, the lamps from $-n$ to 0 will be traversed twice. Therefore, the initial setting of lamps on the first pass can be re-done on the second pass. The asymptotic class of the ray includes all the different initial settings that become the same final setting when the lamplighter moves in its final direction. Thus, the infinite lamp stand of a ray is actually an invariant of its asymptotic class.

Notice that such a ray that has the lamplighter go from 0 to $-n$ and then in the positive direction forever is in $\partial_\infty L_2^+$, but is close in the visual boundary topology to rays in $\partial_\infty L_2^-$ that have the lamplighter only move in the negative direction and agree on the initial settings of the lamps 0 through $-n$. The fact that these initial settings can be made arbitrary within the asymptotic equivalence class gives us a large subset of $\partial_\infty L_2^-$ that is contained in a neighborhood of *any* ray in $\partial_\infty L_2^+$ where the lamplighter moves in the negative direction for a long time before eventually moving in the positive direction forever.

Thus, there exist distinct elements of $\partial_\infty L_2^\pm$ whose neighborhoods always intersect, and that intersection is a subset of $\partial_\infty L_2^\mp$. Therefore $\partial_\infty L_2$ is not Hausdorff. Recall that both $\partial_\infty L_2^\pm$ are punctured Cantor sets under the subspace topology. So, while the subspace topologies of these “halves” are Hausdorff, they are not compact. The full visual boundary $\partial_\infty L_2$ is, however, compact, since these troublesome open sets that intersect both $\partial_\infty L_2^\pm$ “fill” the punctures with open sets in the opposite half.

3.2. The visual boundary as a subset of the horoboundary. We now show that there is a natural injection from the non-Hausdorff $\partial_\infty L_2 = \partial_\infty DL(2, 2)$ into $\partial_h L_2$. Since $\partial_h L_2$ is Hausdorff, this injection is non-continuous.

Lemma 3.1 (Lemma 8.18(1) in Chapter II.8 of [1]). *Let γ be a geodesic ray in $DL(2, 2)$ based at the identity. Then the sequence of points $(\gamma(n))$ defines a horofunction \mathfrak{b}^γ .*

The horofunction \mathfrak{b}^γ is called the *Busemann function* associated to γ . In a $CAT(0)$ space, the Busemann functions of two rays are equal if and only if those two rays are asymptotic. Even though $DL(2, 2)$ is not $CAT(0)$, the same is true in our case.

Lemma 3.2. *Let γ, γ' be geodesic rays in the Cayley graph of L_2 based at the identity. The Busemann functions \mathfrak{b}^γ and $\mathfrak{b}^{\gamma'}$ are equal if and only if γ and γ' are asymptotic to each other.*

Proof. Recall that asymptotic rays in $DL(2, 2)$ eventually merge. Thus, the Busemann functions of asymptotic rays are equal.

Now suppose that γ and γ' are *not* asymptotic to each other. Let $\alpha \in [\gamma], \alpha' \in [\gamma']$ (i.e. α is in the asymptotic equivalence class of γ) so that α and α' have maximal shared initial segment. Say that this shared initial segment has length k . Let $x = \alpha(k+1)$. Notice that by definition, $\mathfrak{b}^\alpha(x) = -(k+1)$. By our choice of α and α' , $\mathfrak{b}^{\alpha'}(x) = -(k-1)$, so $\mathfrak{b}^\alpha \neq \mathfrak{b}^{\alpha'}$. By the proof above of the other direction, $\mathfrak{b}^\gamma = \mathfrak{b}^\alpha$ and $\mathfrak{b}^{\gamma'} = \mathfrak{b}^{\alpha'}$ and we are done. \square

Corollary 3.3. *The relation taking an asymptotic equivalence class of geodesic rays based at the identity to their Busemann functions is an injection of $\partial_\infty L_2$ into $\partial_h L_2$.*

Observation 3.4. *The injection in Corollary 3.3 is not continuous.*

Proof. The continuous injective image of a non-Hausdorff space like $\partial_\infty L_2$ must also be non-Hausdorff, while $C(L_2)$ (and thus its subspace $\partial_h L_2$) is Hausdorff. \square

Recall that the non-Hausdorff property was proved by finding neighborhoods of distinct elements of $\partial_\infty L_2^+$ that always shared elements of $\partial_\infty L_2^-$. Observation 6.2 shows that the restriction of this injection to either of the subspaces of the visual boundary $\partial_\infty L_2^\pm$ is continuous.

4. MODEL HOROFUNCTIONS

In this section, we construct four families of “model” horofunctions, and in §5 we show that these represent all horofunctions.

We break $\partial_h L_2$ into four categories: the Busemann points, the *spine*, the *ribs*, and the two points $\pm H$. The reader may refer to Figure 1 on page 4 to preview a visualization of the boundary, illustrating our choice of terms. To determine which category a sequence (x_n) in L_2 falls into (if it defines a horofunction at all), it turns out we need only consider whether $H(x_n)$ approaches an integer or $\pm\infty$, and whether the union over all lit lamps in the sequence is bounded above or below.

4.1. The Spine. Fix $l \in \mathbb{Z}$, and let $(s_n^l), n \in \mathbb{N}$, be the sequence in L_2 having lamps at $\pm n$ lit, all others unlit, and $H(s_n^l) = l$.

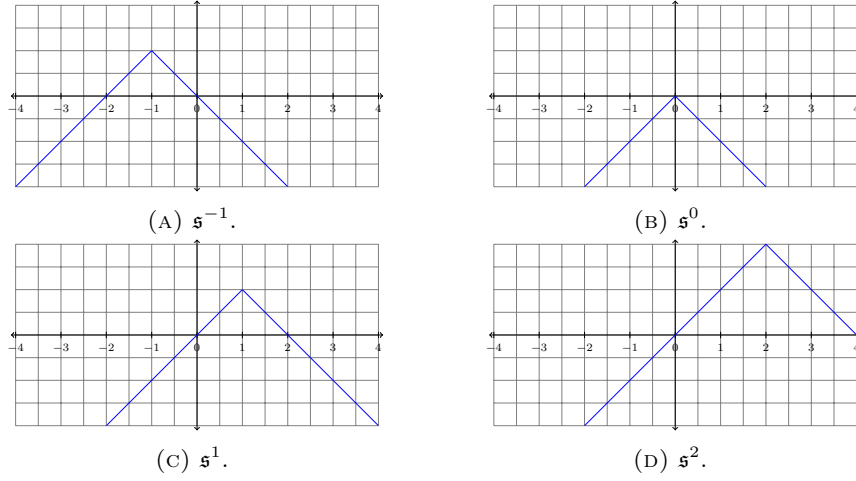
Applying Lemma 2.2 with $A = -n$, $B = n+1$, and $C = |l|$, we have $d(s_n^l, id) = 4n+2-|l|$. Given any $g \in L_2$, take $n > \max\{-m(g), M(g), |H(g)|, |l|\}$ and apply Lemma 2.2 to obtain $d(s_n^l, g) = 4n+2-|l-H(g)|$. By Equation 1, the horofunction is

$$(2) \quad \mathfrak{s}^l(g) = h_{s_n^l}(g) = |l| - |l - H(g)|.$$

We call this the *spine horofunction at height l* . For a given l , this is a function of only $H(g)$. Figure 4 shows the graphs of \mathfrak{s}^{-1} , \mathfrak{s}^0 , \mathfrak{s}^1 , and \mathfrak{s}^2 , respectively, as functions of height. The spine horofunction at height 0 is $\mathfrak{s}^0 = -|H(g)|$. One can check that the sequence (s_n^n) yields $H(g)$ and (s_n^{-n}) yields $-H(g)$; and we can see that the spine functions interpolate between the two.

4.2. The Ribs. The rib horofunctions will be parameterized by certain elements of L_2 . There are two subfamilies, corresponding to the $+\infty$ and $-\infty$ direction, and the generating set $\{t, at\}$ creates a slight asymmetry between them. Let $f \in L_2$, and set $l = H(f)$.

First, assume $M(f) < l$ (noting that $M(f)$ may equal $-\infty$). Then consider the sequence $(r_n^{+,f})$, $n \geq l$, in L_2 where the lamps of $r_n^{+,f}$ agree with those of f in each

FIGURE 4. Four “spinal” horofunctions, as functions of height of $g \in L_2$.

position below l , $H(r_n^{+,f}) = l$, lamp n is lit, and no other lamps at positions l or above are lit.

Given $g \in L_2$, take n large enough, and we have:

$$\begin{aligned} d(r_n^{+,f}, id) &= 2((n+1) - \min\{m(f), l, 0\}) - |l| \\ d(r_n^{+,f}, g) &= 2((n+1) - \min\{m(f, g), H(g), l\}) - |l - H(g)| \end{aligned}$$

This yields the (*positive*) *rib horofunction* corresponding to f :

$$\begin{aligned} \mathfrak{r}^{+,f}(g) &= 2(\min\{m(f), l, 0\} - \min\{m(f, g), H(g), l\}) - |l - H(g)| + |l| \\ (3) \quad &= 2(\min\{m(f), l, 0\} - \min\{m(f, g), H(g), l\}) + \mathfrak{s}^l(g) \end{aligned}$$

We can see that if we had chosen an element whose lamps agreed with f below l , but also had lamps in position l or higher lit, defining a sequence similarly would lead us to the same horofunction, since we can always toggle lamps at l or above “for free” with the generator *at*.

Though the set of positive rib horofunctions is discrete, there is some structure to be observed. Given a height l , let R_l^+ be the set of positive rib horofunctions at height l . Each corresponds to an element $f \in L_2$ with $H(f) = l$ and $M(f) < l$. Then the “minimum lit lamp” map $m : L_2 \rightarrow \overline{\mathbb{Z}}$ induces a map $\hat{m}_l : R_l^+ \rightarrow \overline{\mathbb{Z}}$. For $k \in \overline{\mathbb{Z}}$, the cardinality of $\hat{m}_l^{-1}(k)$ is $2^{(l-k-1)}$ if $k < l$, 1 if $k = +\infty$, and 0 otherwise. The set R_l^+ can then be partitioned according to the nonempty preimages, which provides a natural filtration of R_l^+ . Any sequence (r_n) of horofunctions in R_l^+ corresponding to a sequence (f_n) in L_2 with $m(f_n) \rightarrow -\infty$, will approach \mathfrak{s}^l . We make a precise argument for this fact in Observation 6.4.

In the special case that f has no lit lamps, then $M(f) = -\infty$ and $m(f) = +\infty$ and the calculation simplifies. Since $m(f, g) = m(g)$, the only data is the height $l = H(f)$; and we have:

$$\begin{aligned}
\mathfrak{r}^{+,f}(g) &= \mathfrak{r}^{+,l}(g) = 2(\min\{l, 0\} - \min\{m(g), H(g), l\}) - |l - H(g)| + |l| \\
(4) \qquad \qquad &= -2\min\{m(g), H(g), l\} - |l - H(g)| + l
\end{aligned}$$

As indicated in the preceding paragraph, when there are no lit lamps the resulting horofunction $\mathfrak{r}^{+,l}$ is in a sense the farthest positive rib function of height l from the spine, and we think of it as the *rib tip* at height l .

We now turn to the negative rib functions, corresponding to those f that satisfy $m(f) \geq l$ (possibly with $m(f) = +\infty$). Note we use “ \geq ” now, whereas we used “ $<$ ” previously, since the status of the lamp at l does matter in this direction, since using $(at)^{-1}$ will only let us toggle lamps in positions $l-1$ or lower “for free”. One can define a corresponding sequence similarly to the positive direction, except that the lit lamps approach $-\infty$, and calculate the horofunction:

$$\begin{aligned}
\mathfrak{r}^{-,f}(g) &= 2(\max\{M(f, g) + 1, H(g), l\} - \max\{M(f) + 1, l, 0\}) - |l - H(g)| + |l| \\
(5) \qquad \qquad &= 2(\max\{M(f, g) + 1, H(g), l\} - \max\{M(f) + 1, l, 0\}) + \mathfrak{s}^l(g)
\end{aligned}$$

There is a similar simplification in this direction when f has no lit lamps, so that the horofunction depends only on l :

$$\begin{aligned}
\mathfrak{r}^{-,f}(g) &= \mathfrak{r}^{-,l}(g) = 2(\max\{M(g) + 1, H(g), l\} - \max\{l, 0\}) - |l - H(g)| + |l| \\
(6) \qquad \qquad &= 2\max\{M(g) + 1, H(g), l\} - |l - H(g)| - l
\end{aligned}$$

Finally, the set R_l^- of negative rib horofunctions at height l has a structure similar to R_l^+ .

4.3. Busemann Functions. Given a geodesic ray γ with $\gamma(0) = id$, let \mathfrak{b}^γ denote its horofunction. Let $g \in L_2$. As discussed in Definition 2.1 and §3.1, we can define the functions m and M similarly for γ . We either have $\gamma \in \partial_\infty L_2^+$ and $m(\gamma)$ and $m(\gamma, g)$ are defined, or $\gamma \in \partial_\infty L_2^-$ and $M(\gamma)$ and $M(\gamma, g)$ are defined. The formula for \mathfrak{b}^γ depends on the direction of γ , so we use $\mathfrak{b}^{+, \gamma} = \mathfrak{b}^\gamma$ when $\gamma \in \partial_\infty L_2^+$ and $\mathfrak{b}^{-, \gamma} = \mathfrak{b}^\gamma$ when $\gamma \in \partial_\infty L_2^-$, to be clear.

When $\gamma \in \partial_\infty L_2^+$, for n large enough, we apply Lemma 2.2 to obtain:

$$\begin{aligned}
d(\gamma(n), id) &= 2(H(\gamma(n)) - \min\{m(\gamma), 0\}) - H(\gamma(n)) \\
d(\gamma(n), g) &= 2(H(\gamma(n)) - \min\{m(\gamma, g), H(g)\}) + H(g) - H(\gamma(n))
\end{aligned}$$

Thus the Busemann function corresponding to γ is given by

$$(7) \qquad \mathfrak{b}^{+, \gamma}(g) = 2(\min\{m(\gamma), 0\} - \min\{m(\gamma, g), H(g)\}) + H(g)$$

If $\gamma \in \partial_\infty L_2^-$, we can similarly calculate

$$(8) \qquad \mathfrak{b}^{-, \gamma}(g) = 2(\max\{M(\gamma, g) + 1, H(g)\} - \max\{M(\gamma) + 1, 0\}) - H(g)$$

Note that the Busemann horofunctions are obtained from the rib horofunctions by allowing the lamplighter position to approach $+\infty$ or $-\infty$ as appropriate. This is spelled out later in Observation 6.5.

Given any two horofunctions described above, one can find an element g of L_2 on which they disagree. Thus we have the following observation.

Observation 4.1. *The horofunctions \mathfrak{s}^l for $l \in \mathbb{Z}$, $\pm H$, $\mathfrak{r}^{+,f}$ for $f \in L_2$ and $M(f) < H(f)$, $\mathfrak{r}^{-,f}$ for $f \in L_2$ and $m(f) \geq H(f)$, $\mathfrak{b}^{+, \gamma}$, $\mathfrak{b}^{-, \gamma}$, $\gamma \in \partial_\infty L_2$, are all pairwise distinct.*

5. CLASSIFICATION OF HOROFUNCTIONS

We will now prove that the functions referred to in Observation 4.1 constitute all of $\partial_h L_2$.

Definition 5.1. Given a sequence $(g_n) \subset L_2$, we say that the lamp at position k in the lamp stands of these elements *stabilizes* if there exists $N \in \mathbb{N}$ such that the lamp in position k for the lamp stand representing g_n has the same status (i.e. on or off) for all $n > N$.

We say that the lamp at position k is *flickering* if it does not stabilize.

Definition 5.2. We say that sequence (g_n) of elements of L_2 is *right stable* if there exists $N \in \mathbb{N}$ and $M \in \mathbb{Z}$ such that for all $k > M$, the lamp at position k for the lamp stand representing g_n has the same status (i.e. on or off) for all $n > N$. That is, a sequence is right stable if the set of positions of its flickering lamps (should any exist) has a maximum.

We define *left stable* similarly.

Observation 5.3. *If a sequence $(g_n) \subset L_2$ is not right stable, then there exists a subsequence (g_{n_k}) such that the sequence $(M(g_{n_k}))$ is increasing without bound.*

Similarly, if a sequence $(g_n) \subset L_2$ is not left stable, then there exists a subsequence (g_{n_k}) such that the sequence $(m(g_{n_k}))$ is decreasing without bound.

Proof. If the sequence is not right stable, then $\sup\{M(g_n) \mid n \in \mathbb{N}\} = +\infty$ since if this supremum were a finite value $M_0 \in \mathbb{Z}$, then by setting $N = 0$ and $M = M_0$, the sequence would satisfy the definition for being right stable. The existence of the desired subsequence is then guaranteed.

The proof when the sequence is not left stable is similar. \square

Lemma 5.4. *Suppose that a sequence $(g_n) \subset L_2$ with $H(g_n) \rightarrow l \in \mathbb{Z} \cup \{+\infty\}$ is left stable. If (g_n) is associated with some horofunction h_{g_n} , then the set of positions of its flickering lamps (should any exist) has a minimum of at least l .*

Proof. If (g_n) has no flickering lamps, then we are done. So assume the sequence has some flickering lamps, and let $k \in \mathbb{Z}$ be the minimum position of a flickering lamp. Suppose for contradiction that $k < l$.

Let $y \in L_2$ such that $H(y) = k$, y agrees with the stabilization of lamps of (g_n) on the positions $k - 1$ and below, and the lamp at position k is off. Let $x \in L_2$ be exactly as y , except that $H(x) = k + 1$. Let n be sufficiently large so that the lamps at positions $k - 1$ and below of g_n have achieved their eventual status and $H(g_n) > k$.

Suppose the lamp at position k is lit in the lamp stand for g_n . In Lemma 2.2, when computing $d(g_n, x)$, $C = H(g_n) - (k + 1)$, but when computing $d(g_n, y)$, $C = H(g_n) - k$, while the values for A and B remain the same (in this case, $A = k$ for both). Thus $d(g_n, x) = d(g_n, y) + 1$.

Now suppose the lamp at position k is *not* lit in the lamp stand for g_n . Using Lemma 2.2 again, when computing $d(g_n, x)$, $A = k + 1$, $C = H(g_n) - (k + 1)$, while when computing $d(g_n, y)$, $A = k$, $C = H(g_n) - k$, and B remains the same. In this case, we have $d(g_n, x) = d(g_n, y) - 1$.

By Equation 1,

$$h_{g_n}(x) - h_{g_n}(y) = \lim_{n \rightarrow \infty} d(g_n, x) - d(g_n, y)$$

which by the above, does not exist. But we assumed h_{g_n} exists. Hence, our assumption that $k < l$ is incorrect, and we have the desired result. \square

Lemma 5.5. *Suppose that a sequence $(g_n) \subset L_2$ with $H(g_n) \rightarrow l \in \{-\infty\} \cup \mathbb{Z}$ is right stable. If (g_n) is associated with some horofunction $h = h_{g_n}$, then for every $k \geq l$, the lamp at position k stabilizes.*

Proof. The proof for this lemma is the same as for Lemma 5.4. The asymmetry in the inequalities (one is strict, while the other is not) comes from the asymmetry of our generating set (including at but not ta). \square

Lemma 5.6. *Suppose that a sequence $(g_n) \subset L_2$ is both left and right stable and that $H(g_n) \rightarrow l \in \mathbb{Z}$. If (g_n) is associated with some horofunction h_{g_n} , then there is $g \in L_2$ such that $g_n \rightarrow g$ (i.e. the sequence is eventually constant), and h_{g_n} is associated to the image of g in $\overline{L_2}$.*

Proof. By Lemmas 5.4 and 5.5, all the lamps in (g_n) stabilize. Since it is stable on both sides, we in fact have the existence of some $N \in \mathbb{N}$ such that the set of lit lamps in g_n is constant for all $n > N$. Since the lamplighter limits to l by hypothesis and since \mathbb{Z} is a discrete set, we have that the sequence (g_n) is eventually constant. \square

Lemma 5.7. *Suppose that a sequence $(g_n) \subset L_2$ is either left or right stable, but not both, and that $H(g_n) \rightarrow l \in \mathbb{Z}$. If (g_n) is associated with some horofunction h_{g_n} , then h_{g_n} is a rib, i.e one of $\mathfrak{r}^{\pm, f}$, $f \in L_2$.*

Proof. We consider the case where the sequence (g_n) is left stable, but not right stable. The other case is similar.

By Lemma 5.4, there exists $N \in \mathbb{N}$ such that the the lamps below position l are stable and $H(g_n) = l$ for all $n > N$. Let \mathfrak{r} be the rib horofunction that matches this stabilization. Set (r_n) to be the model sequence defined in Section 4.2 that generates this horofunction.

By Observation 5.3, we may take a subsequence (g_{n_k}) such that $(M(g_{n_k}))$ is increasing with $M(g_{n_k}) > k$ for all k . Choose a subsequence (r_{n_k}) of our model sequence such that $M(r_{n_k}) = M(g_{n_k})$.

Let $x \in L_2$. Choose $K \in \mathbb{N}$ such that $K > \max\{|l|, |M(x)|, |H(x)|\}$, and let $k > K$.

Let A, B, C be as in Lemma 2.2 for the computation of $d(g_{n_k}, x)$ and let A', B', C' be as in Lemma 2.2 for the computation of $d(r_{n_k}, x)$. Notice that $A = A'$ since the lamp stands for g_{n_k} and r_{n_k} are the same below the position $H(g_{n_k}) = H(r_{n_k})$, $B = M(g_{n_k}) + 1 = M(r_{n_k}) + 1 = B'$ by our choice of K , and $C = C'$ since $H(g_{n_k}) = H(r_{n_k})$. Thus, $d(g_{n_k}, x) = d(r_{n_k}, x)$.

For $x = id$, we have that $d(g_{n_k}, id) = d(r_{n_k}, id)$. Hence, $h_{g_{n_k}} = h_{r_{n_k}}$ and so therefore $h_{g_n} = \mathfrak{r}$. \square

Lemma 5.8. *Suppose that a sequence $(g_n) \subset L_2$ is neither left nor right stable and that $H(g_n) \rightarrow l \in \mathbb{Z}$. If (g_n) is associated with some horofunction h_{g_n} , then $h_{g_n} = \mathfrak{s}^l$.*

Proof. Suppose that there exists a subsequence (g_{n_k}) such that for all $N \in \mathbb{N}$ there exists $K_N \in \mathbb{N}$ such that for all $k > K_N$ we have that $M(g_{n_k}) > N$ and $m(g_{n_k}) < -N$. Then let $x \in L_2$, and let $N \in \mathbb{N}$ such that $N > \max\{|M(x)|, |m(x)|, |l|\}$. Let $K = \max\{K_N, K_l\}$ where K_N is as given above and K_l is an integer such that for all $k > K_l$, $H(g_{n_k}) = l$ (recall that $H(g_n) \rightarrow l$ and the integers are a discrete set).

Let $k > K$. Then by choice of N and definition of K and using Lemma 2.2, $d(g_{n_k}, x) = 2(M(g_{n_k}) + 1 - m(g_{n_k})) - |l - H(x)|$ and specifically $d(g_{n_k}, id) = 2(M(g_{n_k}) + 1 - m(g_{n_k})) - |l|$. Thus, by Equation 1 $h_{g_{n_k}} = \mathfrak{s}^l(x)$, and we are done.

Now suppose that such a subsequence does *not* exist. By Observation 5.3, since (g_n) is not left stable, there exists a subsequence (g_{n_i}) such that $m(g_{n_i}) < -i$ for all i and $(m(g_{n_i}))$ is decreasing. Also by Observation 5.3, since (g_n) is not right stable, there exists a subsequence (g_{n_j}) such that $M(g_{n_j}) > j$ for all j and $(M(g_{n_j}))$ is increasing. Since these are both subsequences of (g_n) , both give rise to horofunctions, and $h_{g_{n_i}} = h_{g_{n_j}} = h_{g_n}$.

Notice that the subsequence (g_{n_i}) must be right stable, otherwise we would be able to find a subsequence as in the first part of the proof. Similarly, the subsequence (g_{n_j}) must be left stable.

By Lemma 5.7, $h_{g_{n_i}}$ is equal to one of the rib examples with stable component above the lamplighter. But also by Lemma 5.7, $h_{g_{n_j}}$ is equal to one of the rib examples with stable component *below* the lamplighter. By inspecting Equations 3 and 5, we see that these two horofunctions cannot be equal, so h_{g_n} does not exist. \square

Lemma 5.9. *Suppose that a sequence $(g_n) \subset L_2$ is left stable and $H(g_n) \rightarrow +\infty$. If (g_n) is associated with some horofunction h_{g_n} , then h_{g_n} is equal to a Busemann function \mathfrak{b}^γ with $[\gamma] \in \partial_\infty L_2^+$.*

Proof. By Lemma 5.4, there are no flickering lamps in (g_n) , so consider the infinite lamp stand of the stabilization of lamps in (g_n) . Since the sequence is left stable, if there are any lamps lit in this infinite lamp stand, there is a minimum such lamp. Thus, there exists $[\gamma] \in \partial_\infty L_2^+$ with infinite lamp stand equal to this stabilization.

Take a subsequence (g_{n_k}) such that for every positive integer K , for all $k > K$ the lamps at positions at most K in the lamp stand for g_{n_k} have achieved their eventual status and $H(g_{n_k}) > K$.

Let $x \in L_2$. Set K to be sufficiently large so that $K \geq \max\{m(x), M(x), H(x)\}$, and for the finite values of $m(\gamma)$ and $m(\gamma, x)$, $K \geq \max\{m(\gamma), m(\gamma, x)\}$ as well.

Let $k > K$. Assume that h_{g_n} exists (and is therefore equal to $h_{g_{n_k}}$) and use Lemma 2.2 and Equation 1:

$$\begin{aligned} h_{g_{n_k}}(x) &= \lim_{n_k \rightarrow \infty} 2(\max\{M(g_{n_k}, x) + 1, H(g_{n_k}), H(x)\} \\ &\quad - \min\{m(g_{n_k}, x), H(x)\}) - (H(g_{n_k}) - H(x)) \\ &\quad - [2(\max\{M(g_{n_k}) + 1, H(g_{n_k}), 0\} - \min\{m(\gamma), 0\}) - H(g_{n_k})] \end{aligned}$$

Notice that if $\max\{M(g_{n_k}), M(g_{n_k}, x)\} > H(g_{n_k})$, then since $H(g_{n_k}) > M(x)$, we have that $M(g_{n_k}, x) = M(g_{n_k})$. Since $H(g_{n_k}) \geq \max\{H(x), 0\}$, we have that $\max\{M(g_{n_k}, x) + 1, H(g_{n_k}), H(x)\} = \max\{M(g_{n_k}) + 1, H(g_{n_k}), 0\}$. Therefore,

$$h_{g_{n_k}}(x) = \lim_{n_k \rightarrow \infty} 2(\min\{m(\gamma), 0\} - \min\{m(g_{n_k}, x), H(x)\}) + H(x)$$

Now notice that if $m(g_{n_k}) < 0$ or $m(\gamma) < 0$, then $m(\gamma) = m(g_{n_k})$. Similarly, if $m(g_{n_k}, x) < H(x)$ or $m(\gamma, x) < H(x)$, since $H(x) < K$, then $m(g_{n_k}, x) = m(\gamma, x)$. So by Equation 7 and the above, $h_{g_n} = \mathfrak{b}^\gamma$. \square

Lemma 5.10. *Suppose that a sequence $(g_n) \subset L_2$ is not left stable and $H(g_n) \rightarrow +\infty$. If (g_n) is associated with some horofunction h_{g_n} , then $h_{g_n} = H$, the height function.*

Proof. By Observation 5.3, (g_n) has a subsequence (g_{n_i}) such that $(m(g_{n_i}))$ is decreasing with $m(g_{n_i}) < -i$ for all i . We still have $H(g_{n_i}) \rightarrow +\infty$, so we can further take a subsequence (g_{n_k}) such that for all k , $m(g_{n_k}) < -k$ and $H(g_{n_k}) > k$.

Let $x \in L_2$, let $K = \max\{M(x), |m(x)|, |H(x)|\}$, and consider $k > K$. By Lemma 2.2, there exists $B \in \mathbb{Z}$ such that

$$d(g_{n_k}, x) = 2(B - m(g_{n_k})) - |H(g_{n_k}) - H(x)|$$

and

$$d(g_{n_k}, id) = 2(B - m(g_{n_k})) - |H(g_{n_k})|.$$

Thus,

$$h_{g_{n_k}}(x) = \lim_{n_k \rightarrow \infty} |H(g_{n_k})| - |H(g_{n_k}) - H(x)| = H(x)$$

\square

Lemma 5.11. *Suppose that a sequence $(g_n) \subset L_2$ is right stable and $H(g_n) \rightarrow -\infty$. If (g_n) is associated with some horofunction h_{g_n} , then h_{g_n} is equal to a Busemann function \mathfrak{b}^γ with $[\gamma] \in \partial_\infty L_2^-$.*

Proof. As in the proof of Lemma 5.9, but the Busemann function will have the lamplighter at $-\infty$ instead of $+\infty$. \square

Lemma 5.12. *Suppose that a sequence $(g_n) \subset L_2$ is not right stable and $H(g_n) \rightarrow -\infty$. If (g_n) is associated with some horofunction h_{g_n} , then $h_{g_n} = -H$, the negation of the height function.*

Proof. Similar to the proof of Lemma 5.10. \square

Theorem 5.13. *Suppose that a sequence $(g_n) \subset L_2$ has $(H(g_n))$ converging to some value $l \in \mathbb{Z} \cup \{\pm\infty\}$. If (g_n) is associated with some horofunction h_{g_n} and $g \in L_2$, then:*

- (1) *If $l \in \mathbb{Z}$ and (g_n) is both left and right stable, then the sequence is eventually a constant value g_0 and h_{g_n} is in the image of L_2 in $\overline{L_2}$.*

$$h_{g_n}(g) = d(g, g_0)$$

- (2) *If $l = +\infty$ and (g_n) is left stable, then $h_{g_n} = \mathfrak{b}^\gamma$ for some $[\gamma] \in \partial_\infty L_2^+$.*

$$\mathfrak{b}^{+, \gamma}(g) = 2(\min\{m(\gamma), 0\} - \min\{m(\gamma, g), H(g)\}) + H(g)$$

- (3) *If $l = -\infty$ and (g_n) is right stable, then $h_{g_n} = \mathfrak{b}^\gamma$ for some $[\gamma] \in \partial_\infty L_2^-$.*

$$\mathfrak{b}^{-, \gamma}(g) = 2(\max\{M(\gamma, g) + 1, H(g)\} - \max\{M(\gamma) + 1, 0\}) - H(g)$$

- (4) *If $l \in \mathbb{Z}$ and (g_n) is neither left nor right stable, then $h_{g_n} = \mathfrak{s}^l$*

$$\mathfrak{s}^l(g) = |l| - |l - H(g)|$$

(5) If $l \in \mathbb{Z}$ and (g_n) is left-but not right-stable, then $h_{g_n} = \mathfrak{r}^{+,f}$ for some $f \in L_2$.

$$\mathfrak{r}^{+,f}(g) = 2(\min\{m(f), l, 0\} - \min\{m(f, g), H(g), l\}) + \mathfrak{s}^l(g)$$

(6) If $l \in \mathbb{Z}$ and (g_n) is right-but not left-stable, then $h_{g_n} = \mathfrak{r}^{-,f}$ for some $f \in L_2$.

$$\mathfrak{r}^{-,f}(g) = 2(\max\{M(f, g) + 1, H(g), l\} - \max\{M(f) + 1, l, 0\}) + \mathfrak{s}^l(g)$$

(7) If $l = +\infty$ and (g_n) is not left stable, then $h_{g_n} = H$.

(8) If $l = -\infty$ and (g_n) is not right stable, then $h_{g_n} = -H$.

Proof. If $l \in \mathbb{Z}$, then apply one of Lemmas 5.6, 5.7, or 5.8, as appropriate for the existence of left or right stability. If $l = +\infty$, then apply either Lemma 5.9 or 5.10, depending on the existence of left stability. If $l = -\infty$, then apply either Lemma 5.11 or 5.12, depending on the existence of right stability. \square

Lemma 5.14. Suppose for a sequence $(g_n) \subset L_2$, $(H(g_n))$ does not converge in $\mathbb{Z} \cup \{\pm\infty\}$. Then (g_n) is not associated with a horofunction.

Proof. By our hypotheses, (g_n) has subsequences (g_{n_i}) and (g_{n_j}) such that $(H(g_{n_i}))$ and $(H(g_{n_j}))$ converge in $\mathbb{Z} \cup \{\pm\infty\}$, but to distinct values. By Theorem 5.13 and Observation 4.1, since these limits are distinct, $h_{g_{n_i}} \neq h_{g_{n_j}}$. Thus h_{g_n} cannot exist. \square

Corollary 5.15. Let $h \in \overline{L_2}$, and choose a sequence $(g_n) \subset L_2$ such that $h = h_{g_n}$. Then $(H(g_n))$ converges to some value $l \in \mathbb{Z} \cup \{\pm\infty\}$, and h_{g_n} can be categorized as in Theorem 5.13

6. TOPOLOGY OF THE HOROFUNCTION BOUNDARY

The topology of $\partial_h L_2$ is the topology of uniform convergence on compact sets. The standard basis is the collection of sets of the form

$$B_K(h, \epsilon) = \{h' \in \partial_h L_2 \mid |h(x) - h'(x)| < \epsilon \text{ for all } x \in K\}$$

where $K \subset L_2$ is compact and $\epsilon > 0$. By restricting to $0 < \epsilon < 1$, we obtain an equivalent basis. Since the minimum distance between distinct points in L_2 is 1, we may use the following sets as a basis:

$$B_K(h) = \{h' \in \partial_h L_2 \mid h(x) = h'(x) \text{ for all } x \in K\}$$

where $K \subset L_2$ is finite. Notice that pointwise convergence implies convergence in our topology since compact sets of L_2 are finite.

With the explicit descriptions of the horofunctions found in Section 4, we can establish the accumulation points of $\partial_h L_2$. We begin by recalling that since $\mathfrak{s}^l(g) = |l| - |l - H(g)|$, we have

Observation 6.1. $\mathfrak{s}^l \rightarrow \pm H$ as $l \rightarrow \pm\infty$.

Observation 6.2. The injective map that takes elements of $\partial_\infty L_2^+$ to their Busemann functions in $\partial_h L_2$ is continuous, and the same is true of $\partial_\infty L_2^-$.

Contrast this result with Observation 3.4, which states that the injection of the union of these two sets into the horofunction boundary is not continuous. Recall that the obstruction to continuity was the non-Hausdorff property, which was proved by finding neighborhoods of distinct elements of $\partial_\infty L_2^+$ that always shared elements of $\partial_\infty L_2^-$.

Proof. Let $[\gamma] \in \partial_\infty L_2^+$, and consider $B_K(\mathfrak{b}^\gamma)$ for some finite $K \subset L_2$. Let $M = \max\{M(g), H(g) \mid g \in K\}$, and let $k \in \mathbb{Z}$ such that $k > M + 2|m(\gamma)|$ if $m(\gamma) < 0$ or $k > M$ otherwise. Consider the set

$$B_{[0,k]}([\gamma], \epsilon) = \{\gamma' \in \partial_\infty L_2^+ \mid \sup\{d(\gamma(x), \gamma'(x)) \mid x \in [0, k]\} < \epsilon\}$$

for $0 < \epsilon < 1$. In [6, Observation 4.1], the authors observe that $B_{[0,k]}([\gamma], \epsilon)$ is an open set in $\partial_\infty L_2^+$. Notice that if $\gamma' \in B_{[0,k]}([\gamma], \epsilon)$, then the lamp stands of γ and γ' agree on all lamps at positions M or below. Thus, by Equation 7, $\mathfrak{b}^\gamma(g) = \mathfrak{b}^{\gamma'}(g)$ for all $g \in K$. Therefore, $\mathfrak{b}^{\gamma'} \in B_K(\mathfrak{b}^\gamma)$ for all $\gamma' \in B_{[0,k]}([\gamma], \epsilon)$, and so our injection is continuous.

The proof for the injection of $\partial_\infty L_2^-$ is similar. \square

The topology of each of these sets is a punctured Cantor set, but in $\partial_h L_2$ these punctures are “filled” by the height function and its negative, as we now show.

Observation 6.3. *If $([\gamma_n]) \subset \partial_\infty L_2^+$ with $m(\gamma_n) \rightarrow -\infty$, then $\mathfrak{b}^{\gamma_n} \rightarrow H$. Similarly, if $([\gamma_n]) \subset \partial_\infty L_2^-$ with $M(\gamma_n) \rightarrow +\infty$, then $\mathfrak{b}^{\gamma_n} \rightarrow -H$.*

Proof. Let $([\gamma_n]) \subset \partial_\infty L_2^+$ with $\lim m(\gamma_n) = -\infty$. By Equation 7,

$$\mathfrak{b}^{\gamma_n}(g) = 2(\min\{m(\gamma_n), 0\} - \min\{m(\gamma_n, g), H(g)\}) + H(g)$$

Fix g and take n large enough so that $m(\gamma_n) < \min\{0, m(g), H(g)\}$, then

$$\mathfrak{b}^{\gamma_n}(g) = 2(m(\gamma_n) - m(\gamma_n)) + H(g) = H(g)$$

Thus, $\mathfrak{b}^{\gamma_n} \rightarrow H$. The other proof is similar. \square

For a given $l \in \mathbb{Z}$, the following observation remarks that the spine is an accumulation point of the positive and negative rib functions. The proofs are calculations similar to those in Observation 6.3.

Observation 6.4. *Let $(f_n^l) \subset L_2$ be a sequence satisfying $M(f_n^l) < H(f_n^l) = l$ and $m(f_n^l) \rightarrow -\infty$ as $n \rightarrow \infty$. Then $\mathfrak{r}^{+, f_n^l} \rightarrow \mathfrak{s}^l$.*

Similarly, if $(f_n^l) \subset L_2$ is a sequence satisfying $m(f_n^l) \geq H(f_n^l) = l$ and $M(f_n^l) \rightarrow \infty$ as $n \rightarrow \infty$, then $\mathfrak{r}^{-, f_n^l} \rightarrow \mathfrak{s}^l$.

Finally, the ribs accumulate to Busemann functions:

Observation 6.5. *For a geodesic ray γ , with $\gamma(0) = id$, set $f_n = \gamma(n)$. If $\gamma \in \partial_\infty L_2^+$, then for large enough n , each f_n defines \mathfrak{r}^{+, f_n} and $\mathfrak{r}^{+, f_n} \rightarrow \mathfrak{b}^{+, \gamma}$. If $\gamma \in \partial_\infty L_2^-$, then for large enough n each f_n defines \mathfrak{r}^{-, f_n} and $\mathfrak{r}^{-, f_n} \rightarrow \mathfrak{b}^{-, \gamma}$.*

Proof. We consider the $\gamma \in \partial_\infty L_2^+$ case. For large enough n , each f_n satisfies the requirements for defining \mathfrak{r}^{+, f_n} . Let $g \in L_2$ be given, and consider Equations 3 and 7. Again for large enough n , $m(f_n) = m(\gamma)$ and $m(f_n, g) = m(\gamma, g)$. Thus

$$\mathfrak{r}^{+, f_n} - \mathfrak{b}^{+, \gamma} = \mathfrak{s}^{H(f_n)}(g) - H(g) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

\square

With Observations 6.1, 6.3, 6.4, and 6.5, we have the picture of the horofunction boundary illustrated in Figure 1 in the introduction.

7. ACTION OF L_2 ON THE HOROFUNCTION BOUNDARY

We now conclude with a few comments about the action of L_2 on $\partial_h L_2$.

An isometric action of a group G on a metric space (X, d) with base point b can be extended to the horofunction boundary $\partial_h X$ in the following way: For $g \in G$ and $(y_n) \subset X$ giving rise to a horofunction, we have that

$$g \cdot h_{y_n}(x) = h_{g \cdot y_n}(x) = \lim_{n \rightarrow \infty} d(g \cdot y_n, x) - d(g \cdot y_n, b)$$

In our setting, the action of L_2 on itself is by left multiplication. We compose lamp stands $g_1 \cdot g_2$ by starting with the lamp stand for g_1 and having the lamplighter move and toggle lamps as in g_2 , but from a starting position of $H(g_1)$ rather than 0.

Observation 7.1. *Let $g \in L_2$, $h \in \overline{L_2}$, and choose $(g_n) \subset L_2$ such that $h = h_{g_n}$. Then $H(g \cdot g_n) \rightarrow H(g) + \lim H(g_n)$, where for $k \in \mathbb{Z}$, $\pm\infty + k$ is understood to mean $\pm\infty$. Also, $(g \cdot g_n)$ is left (resp. right) stable, iff (g_n) is left (resp. right) stable.*

Proof. These statements all follow from the fact that the lamp stand for g has only finitely many lit lamps and the lamplighter at a finite position. \square

Corollary 7.2. *Each of the categories of horofunctions in $\overline{L_2}$ described in Theorem 5.13 is invariant under the action of L_2 .*

Proof. This result follows from Observation 7.1 and Corollary 5.15. \square

Interestingly, this implies the following:

Corollary 7.3. *The height function H and its negation are global fixed points of the action of L_2 on $\partial_h L_2$.*

We now consider the action of L_2 on each of the other categories of horofunctions.

Let $g \in L_2$. The action of g on $\partial_\infty L_2$ is described in [6, §3.4 and §4.6]. If $H(g) \neq 0$, then the action of g on $\partial_\infty L_2$ has two fixed points, which are given the notation g^∞ and $g^{-\infty}$ in [6]. If $H(g) > 0$, then $g^\infty \in \partial_\infty L_2^+$ and $g^{-\infty} \in \partial_\infty L_2^-$. Otherwise, the reverse is true. In the topology of $\partial_\infty L_2$, the action of g has north-south dynamics with attractor g^∞ and repeller $g^{-\infty}$. Recall that in $\partial_\infty L_2$, the punctures in the two Cantor sets are “filled” by points from the opposite Cantor set, while in $\partial_h L_2$, these punctures are filled by H and $-H$. Thus, in the horofunction boundary we see similar dynamics with the visual boundary, except it occurs on the separate sets of $\partial_\infty L_2^+ \cup \{H\}$ and $\partial_\infty L_2^- \cup \{-H\}$.

Observation 7.4. *For $g \in L_2$ with $H(g) \neq 0$, the action of g on $\partial_h L_2$ has four fixed points: $H, -H, \mathfrak{b}^{g^\infty}, \mathfrak{b}^{g^{-\infty}}$. The action of g has north-south dynamics on $\partial_\infty L_2^+ \cup \{H\}$ with poles H and either g^∞ or $g^{-\infty}$ (whichever is in the set) and also on $\partial_\infty L_2^- \cup \{-H\}$ with poles $-H$ and either g^∞ or $g^{-\infty}$ (whichever is in the set). The point g^∞ is always an attractor and the point $g^{-\infty}$ is always a repeller. If $H(g) > 0$, then H is an attractor and $-H$ is a repeller. If $H(g) < 0$, then these roles are reversed.*

For a spinal horofunction $\mathfrak{s}^l \in \partial_h L_2$, $l \in \mathbb{Z}$, the action of g on \mathfrak{s}^l is given by $g \cdot \mathfrak{s}^l = \mathfrak{s}^{H(g)+l}$.

We see similar behavior on the ribs of $\partial_h L_2$ in that the l value is translated by the height of the group element, but there is also additional structure in this

case: Let $g \in L_2$ and let $f \in L_2$ such that $\mathfrak{r}^{+,f}$ exists (i.e. $M(f) < H(f)$). Then $g \cdot \mathfrak{r}^{+,f} = \mathfrak{r}^{+,g\bar{f}}$ where $\bar{g\bar{f}}$ has the lamp stand for gf but with all of the lamps at position $H(gf) = H(g) + H(f)$ and above switched off. Note that g acts as a bijection from $R_{H(f)}^+$ to $R_{H(g)+H(f)}^+$.

Notice that if $m(g) \neq H(g) + m(f)$, then

$$m(\bar{g\bar{f}}) = \min\{H(g) + m(f), m(g)\}$$

Using the notation in Section 4.2, the above yields the following description of the action on rib horofunctions that are “close” to the spine.

Observation 7.5. *Let $g \in L_2$ and $l \in \mathbb{Z}$. Let $k < l$ such that $H(g) + k < m(g)$. The action of g on $\partial_h L_2$ restricted to the subset $\hat{m}_l^{-1}(k)$ of R_l^+ is a bijection onto the subset $\hat{m}_{H(g)+l}^{-1}(H(g) + k)$ of $R_{H(g)+l}^+$*

Corollary 7.6. *Let $g \in L_2$ such that $H(g) = 0$ and let $l \in \mathbb{Z}$. If $k < \min\{m(g) - H(g), l\}$, then the subset $\hat{m}_l^{-1}(k)$ of R_l^+ is invariant under the action of g .*

The action on such a rib $\mathfrak{r}^{+,f}$ leaves $m(f)$ and $H(f)$ fixed, but changes the status of lamps between those positions. This gives a permutation on the set $m_l^{-1}(k)$.

The similar statements also hold for negative ribs.

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